An Alternative Approach to Capital Investment Appraisal

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Abstract:

The paper presents an alternative and original method for establishing the present worth and feasibility of a capital investment where the underlying parameters of interest/discount rate, cash flows and investment lifespan are uncertain. The method, based on Markov chains, complements existing and useful practices such as sensitivity analysis, Monte Carlo simulation, Hillier style probabilistic analysis and fuzzy sets. For the same underlying assumptions, the results of this alternative approach are the same as for these existing approaches, but the approach provides additional insight into discounted cash flow (DCF) analysis under uncertainty. The method will be found useful to persons doing investment analysis and looking at the risks associated with investment.

Keywords:

Markov chains; discounted cash flow; probability; feasibility; present worth
Introduction

The confidence with which investors can place on the outcome of any discounted cash flow analysis depends on an acknowledgment of the uncertainty which exists in the assumed analysis parameters - the interest/discount rate, cash flows and investment lifespan. Conventionally, variability in these assumptions is looked at through approaches such as sensitivity analysis (including the related 'what if' analysis), Monte Carlo simulation, Hillier style probabilistic analysis and fuzzy sets (Carmichael and Balatbat, 2008a). Conclusions are then drawn on the analysis output, namely present, annual or future worth, internal rate of return, payback period and benefit-cost ratio.

The method proposed in this paper complements, and does not replace, these existing approaches. For the same underlying assumptions, the same results are obtained as in these existing approaches. However, the approach provides additional insight to discounted cash flow (DCF) analysis under uncertainty. The method will be found useful to persons doing investment analysis and looking at the risks associated with investment.

The paper's method is based on using Markov chains to model an investment. In particular, states representing different combinations of investment parameters (interest/discount rate, cash flows and investment lifespan) and transitions between states are defined. This leads to the calculation of the probability of being in any state. With each state representing a different present worth outcome, it is then possible to calculate investment feasibility and expected present worth of the investment.

For definiteness, reference in the calculations is to present worth, but the approach is equally applicable to annual worth, future worth, and benefit-cost ratio as measures of an investment's feasibility. Indicators of internal rate of return and payback period follow as a consequence.

The paper first outlines some necessary feasibility and Markov chain theory and then develops the approach through an example. It is seen that the concept of feasibility provides a unifying thread.

Background

The paper provides an original contribution to established investment analysis practices. While Markov chains have been around for many years and have been used for a wide array of applications - marketing, finance, advertising and so on, and the literature is very large, no one appears to have looked at investment analysis in the way this paper does.

Richardson (1973) looks at regional investment with savings as states, and with fixed investment being an absorbing state. Finardi (1989) examines the properties of the Markov transition matrix in the context of perishable and durable goods. Cyert et al. (1962), Barkman (1977), Corcoran (1978), Wort and
Zumwalt (1985) and Betancourt (1999) among others use ageing accounts receivable or payments due as states. Zannetos (1962) and Ijiri and Kaplan (1970) use asset remaining life as states, with transitions reflecting depreciation, and costs attached to this depreciation.

Howard (1960) allows for discounting of future rewards attached to transitions. The states chosen are various but not related to discount rate, cash flow or investment lifespan. Chorn (2004) looks at exploration investment opportunity with states of drill, hold and exit, and transition rewards as net present value and reserve replacement. Bierman (1966), when looking at bond refunding, uses present values of savings as rewards and interest rates as states. Shank (1971) and Corcoran and Leininger (1973) attach costs to the transitions, and respectively adopt states related to earnings from trees as they grow, and states related to stages of a production process. See also Shank (1972). Glover and Ijiri (2002) use 'browser' and 'buyer' as their states with costs as payoff in an e-commerce example.

Feasibility

The notion of feasibility provides a unifying thread to the notions of present worth, annual worth, future worth, internal rate of return, payback period and benefit-cost ratio. Feasibility refers to an investment being worthwhile.

In deterministic analyses, feasibility is readily established. However, with the inclusion of uncertainty, feasibility is no longer a single value, but rather takes a range of values, and it is not readily obvious how far the calculations have to be taken.

Restricting the present discussion to present worth, feasibility \( \Phi \) of an investment for the probabilistic case, may be defined as the probability that the present worth (PW) of the cash flows is positive (Carmichael and Balatbat, 2008a,b). That is,

\[
\text{Feasibility, } \Phi = P[PW > 0]
\]

where \( P[\ ] \) denotes the probability of the contained argument. Equivalent expressions can be given for annual worth, future worth, internal rate of return, payback period and benefit-cost ratio. And these feasibility expressions can be related. Internal rate of return can be interpreted as the discount rate at which the feasibility \( \Phi \) becomes acceptable. Payback period can be interpreted as the time at which the feasibility \( \Phi \) becomes acceptable (Carmichael and Balatbat, 2008a,b).

Markov Chain Approach
A Markov chain works in terms of discrete states, and transitions between states over time. The state variables for an investment are chosen in this paper to be the main investment parameters of interest/discount rates, cash flows and lifespans, and together the parameters define a state space (of dimension equal to the number of parameters). The Markovian property implies that future behaviour only depends on the present. A Markovian assumption would appear reasonable for the present analysis; comment is given later on this. For any given investment, some states may correspond to positive present worths, while some may not.

Established Markov chain theory (for example, Howard, 1971; Taylor and Karlin, 1994; Norris, 1998; Yin and Zhang, 2005; Carmichael, 1987) defines probabilities associated with state transitions, here denoted $p_{ij}$, in going from state $i$ to state $j$; $i, j = 1, 2, ..., N$. It follows that

$$\sum_{j=1}^{N} p_{ij} = 1 \quad (1)$$

and

$$0 \leq p_{ij} \leq 1 \quad (2)$$

Define a stochastic transition matrix $P$ with components $p_{ij}$. For the Markov chain, the probabilities $p_{ij}$ are a function of $i$ and $j$ only. The $p_{ij}$ are taken here to be constants; comment is given later on this.

Define $\pi_i$ as the probability of being in state $i$; $i = 1, 2, ..., N$, and as components of the row vector $\pi$. Then, following Howard (1960, 1971),

$$\pi = \pi P \quad (3)$$

with

$$\sum_{i=1}^{N} \pi_i = 1 \quad (4)$$

Equations (3) and (4) represent $N+1$ equations in $N$ unknowns, and can be used to find $\pi_i$, $i = 1, 2, ..., N$.

Feasibility (the probability that the present worth is positive) and expected present worth are calculated from,

$$\Phi = \sum_{i \text{ with positive } PW} \pi_i$$

$$E[PW] = \sum_{i=1}^{N} PW_i \pi_i$$
If a transition diagram (for example, Figure 1) is used to show the states (boxes) and the transitions (arrows) between the states, then equating the inputs and outputs for each state will give Equation (3). In the transition diagrams given elsewhere in this paper, transitions from a state to itself are not shown in order to not clutter the diagrams.

[Figure 1. Example transition diagram.]

**State Choice and Transition Probability Estimation**

The approach requires the analyst to establish what are representative states and what are representative estimates of transition probabilities.

**Example – interest rates.** For interest rates, central banks publish historical movements and there is a large amount of data that is accessible. For example, Table 1 shows some historical data relating to housing loan rates

[Table 1. Standard variable housing loan rate changes over a 10-year period January 1998 - December 2007 (source, RBA 2008).]

For the 120 months of data, the interest rates fluctuate between 6% and 8.5%. Rounding, say, to the nearest 1% interest rate, this gives 4 states - 6%, 7%, 8%, and 9%, where for example 7% represents the interval from 6.55% to 7.45%. Finer subdivision of the interest rate range will give more states, and coarser subdivision will give fewer states. The choice of states is at the discretion of the analyst. The approach remains the same irrespective of the number of states. And considering that the calculations are performed on a spreadsheet, the computational effort doesn't change with the number of states. On the matter of choosing more states, it is noted that the computations do not suffer the curse of dimensionality, rather the number of computations is approximately proportional to the number of states.

The transition probabilities, $p_{ij}$, can be calculated from counting the number of monthly transitions between states and dividing by the total number of transitions, to give a frequency.

**Example – cash flows.** In an example multiple housing unit development, scheduled over forty eight months, an investor estimates that the monthly net cash flow (revenues minus costs) could vary between -$0.8M and $1.7M depending on sales and construction progress; this is separate from an initial capital outlay of $5.1M. In increments of $0.50M, this gives 7 states: -$1.00M, -$0.50M, $0.00M, $0.50M, $1.00M, $1.50M and $2.00M. Finer subdivision of the cash flow range will give more states, and coarser subdivision will give fewer
states. The approach remains the same irrespective of the number of states. The computational effort changes little with number of states. The transition probabilities, $p_{ij}$, might be estimated from looking at sales predictions based on the current market and economy.

**Comment.** These are only example ways by which states might be chosen and transition probabilities estimated. As in any discipline, estimators would combine historical data, analysis, judgment and experience to come up with what are believed to be best estimates. Interest/discount rate estimates would be based on an understanding of likely interest rate movements in the economy, ability to borrow capital etc. Cash flow and lifespan estimates would be based on an understanding of likely investment costs, returns and the investment environment. The estimates will also depend on the time unit or interval chosen.

**Example**

To demonstrate the approach, assume a common investment profile involving an initial outlay with future returns. For definiteness, consider an initial outlay of $10,000, an annual (net positive) cash flow return of $1,000, an interest/discount rate of 5%, and a lifespan of 15 years.

By conventional calculations, the deterministic present worth, discounted payback period and internal rate of return for these values are $0.38 \times 10^3$, 14.2 years, and 5.6% respectively.

Now allow for uncertainty in the investment parameters of interest/discount rate, the future cash flow and the lifespan of the investment. Fluctuations could be anticipated in these parameters. These parameters can be considered singly and in combinations.

For fluctuations only in the interest/discount rate, and assuming that the other investment parameters stay constant, the state is one dimensional and the transition diagram might look something like Figure 2. The states are numbered in the boxes, and estimates of the transition probabilities are given next to the arrows. The actual values chosen here for the transition probabilities serve only to demonstrate the calculations.

Figure 2, for example purposes, anticipates that the discount rate can fluctuate plus 1% and minus 0.5% about the base case of 5%, while the probability of fluctuations outside this range are assumed small and are neglected.

[Figure 2. Fluctuation in interest/discount rate; example.]

Balancing the inputs and outputs from each state (Equation 3) gives
And using Equation (4) and solving gives

\[ \pi = [0.077 \ 0.154 \ 0.256 \ 0.513]. \]

Of the states, states 1, 2 and 3 correspond to positive present worths, and hence the feasibility of the investment

\[ \Phi = \pi_2 + \pi_3 + \pi_4 = 0.487 \]

The expected value and variance of the present worth of the investment are

\[ E[\text{PW}] = 0.077 \times 0.740 + 0.154 \times 0.380 + 0.256 \times 0.038 - 0.513 \times 0.288 \]
\[ = - \$0.023 \times 10^3 \]
\[ \text{Var[PW]} = 0.077 \times 0.763^2 + 0.154 \times 0.403^2 + 0.256 \times 0.061^2 - 0.513 \times 0.265^2 \]
\[ = 0.035 = (\$0.186 \times 10^3)^2 \]

A normal distribution can be used to model the probability distribution of present worth. Hillier (1963, 1969), Wagle (1967) and Tung (1992) among others support using such an approach, however analysts can use alternative distributions if they wish. Then,

\[ P[\text{PW} > 0] = 0.451 \]

which is consistent, subject to the approximations made, with the earlier \( \Phi \) value of 0.487

Using the probabilities of being in any state, \( \pi_i \), it is possible to calculate an expected value and variance of these state values.

\[ \text{Expected value} = 0.077 \times 4.5 + 0.154 \times 5.0 + 0.256 \times 5.5 + 0.513 \times 6.0 = 5.60\% \]
\[ \text{Variance} = 0.077 \times 1.1^2 + 0.154 \times 0.6^2 + 0.256 \times 0.1^2 + 0.513 \times 0.4^2 = 0.23 \]
\[ = (0.48\%)^2 \]

Should it be desired, then this allows a normal or other distribution to be fitted for the discount rate.

Fluctuations in the other parameters (cash flows and lifespans), taken one at a time, are handled similarly. Where more than one of the parameters (interest/discount rates, future cash flow and lifespan) is allowed to fluctuate, then the dimension of the transition diagram would grow in proportion to the number of fluctuating parameters. For example, allowing the interest/discount
rates, future cash flow and lifespan to fluctuate produces a three-dimensional transition diagram, with each dimension corresponding to one of the parameters. Figure 3 shows an example.

[Figure 3. Fluctuation in discount rate, future cash flow and investment lifespan; example.]

Figure 3 anticipates that the discount rate, future cash flow and investment lifespan can all fluctuate by small amounts. As in the previous example, the probability of fluctuations outside these ranges is assumed small and is neglected. Figure 3 also gives estimated example probabilities associated with anticipated movements in the discount rate, future cash flow and investment lifespan. These estimates would be based on an understanding of the likely investment environment. The estimates will also depend on the time unit or interval chosen, with higher probabilities associated with larger time units.

For the Figure 3 case, the state probabilities \( \pi_i, i = 1, 2, \ldots, 8 \) become respectively 0.076, 0.114, 0.102, 0.152, 0.095, 0.143, 0.127 and 0.190. With the last 6 states corresponding to positive present worths, the feasibility of the investment becomes,

\[
\Phi = \sum_{i=3}^{8} \pi_i = 0.809
\]

The expected present worth of the investment is

\[
E[\text{PW}] = -0.076 \times 0.101 - 0.114 \times 0.410 + 0.102 \times 0.889 + 0.152 \times 0.549 \\
+ 0.095 \times 0.380 + 0.143 \times 0.038 + 0.127 \times 1.418 + 0.190 \times 1.041 \\
= \$0.539 \times 10^3
\]

**Comparison methods**

Comparison comment is given here on existing complementary methods of sensitivity analysis, Monte Carlo simulation, and Hillier style probabilistic analysis.

A sensitivity analysis on the original problem would vary the investment parameters of discount rate, future cash flow and investment period by plus/minus small amounts, usually one parameter at a time. For example, consider varying the discount rate by \( \pm 0.5\% \) either side of the assumed deterministic discount rate of 5%. At 4.5%, 5% and 5.5%, the deterministic present worths of the investment are respectively \( \$0.740 \times 10^3 \), \( \$0.380 \times 10^3 \) and \( \$0.038 \times 10^3 \), exactly as used in the Markov chain approach. The sensitivity analysis is showing that as discount rates increase, the feasibility of the investment decreases. The Markov chain approach additionally attaches
probabilities to these different present worths. In effect, the Markov chain approach incorporates the essence of a sensitivity analysis.

Simulation and Hillier style analysis use probability distributions, or moments of these distributions, to model the uncertainty in the investment parameters. For example, consider uncertainty in future cash flow, and let the annual future cash flow be described by a rectangular probability distribution with mean $1,050 and standard deviation $70. States 5 and 7 of Figure 3, together, come closest to this. The methods of simulation and Hillier style analysis give a present worth probability distribution with a mean and standard deviation (for uncorrelated future cash flows) respectively of $0.899 \times 10^3$ and $0.194 \times 10^3$, which is consistent with approximately combining state 5 (PW = $0.380 \times 10^3$) and state 7 (PW = $1.418 \times 10^3$) of the Markov chain approach. It follows that feasibility, which is defined in terms of present worth, is also consistent between the methods.

**Payback Period and Internal Rate of Return**

Indicators of internal rate of return and payback period can be calculated from the above analyses.

**Payback period**

Consider firstly (discounted) payback period. Two dimensional transition diagrams in terms of discount rate and return are used, such as the example transition diagram of Figure 4.

![Figure 4. Fluctuation in discount rate and future cash flow (x10^3); example.]

The lifespan is varied, and E[PW] is calculated for each lifespan, to give an indication of payback period. See for example Figure 5, based on Figure 4 values. Non-discounted payback period can be worked similarly.

![Figure 5. Variation in E[PW] (x10^3) with lifespan.]

As an alternative, in Carmichael and Balatbat (2008a), feasibility in a payback period sense is defined as

$$\Phi = P[PBP < \text{nominated } t]$$

where PBP is payback period, and t is some time. Accordingly, feasibility takes the same shape as the cumulative distribution function for PBP.
For each lifespan in a range of lifespans, $E[PW]$ and $Var[PW]$ are calculated, and a normal distribution is fitted to these. Since

$$P[PBP > \text{nominated } t] = P[PW < 0 \mid \text{nominated } t]$$

then the cumulative distribution function for PBP is obtained from $1 - P[PBP > t]$. Figure 6 shows the resulting (part) cumulative distribution function.

[Figure 6. Part cumulative distribution function for PBP (years), and feasibility plot.]

**Internal rate of return**

For internal rate of return, two-dimensional transition diagrams in terms of return and lifespan are used, for example a slice through Figure 3 corresponding to a discount rate of 5%, that is states 1, 3, 5 and 7. For a range of discount rates, $E[PW]$ may be calculated to give an indication of internal rate of return. See for example Figure 7, based on Figure 3 values.

[Figure 7. Variation in $E[PW]$ ($\times 10^3$) with discount rate.]

As an alternative, in Carmichael and Balatbat (2008a) feasibility in an internal rate of return sense is defined as

$$\Phi = P[IRR > \text{nominated } r]$$

where IRR is internal rate of return, and $r$ is some discount rate. Accordingly, feasibility can be established from the cumulative distribution function for IRR, and is shown in Figure 8 also.

For each discount rate in a range of discount rates, $E[PW]$ and $Var[PW]$ are calculated, and a normal distribution is fitted to these. Then the cumulative distribution function for IRR is obtained from (Hillier, 1963),

$$P[IRR < r] = P[PW < 0 \mid r]$$

Figure 8 shows the resulting (part) cumulative distribution function.

[Figure 8. Part cumulative distribution function for IRR, and feasibility plot.]

**Comment.** Figures 5 to 8 are nonlinear; it is the drawing resolution which makes them appear linear.
Discussion and Conclusion

The above exampled state transition diagrams can be enlarged by incorporating more states or finer divisions between states. The number of states and what they represent is up to the discretion of the analyst. Having more states does no more than increase the amount of computations. The approach doesn't change. All the above numerical computations were readily carried out on a spreadsheet. Increasing the number of states will not make the computations any more difficult, rather only the number of computations will increase.

On the matter of choosing more states, it is noted that the associated computations do not suffer the curse of dimensionality. The number of computations is roughly proportional to the number of states. So, for example, doubling the number of states approximately doubles the associated computations.

To keep the computations at a manageable size, some state truncation might be considered. States, which are anticipated to have a low probability of occurrence, could be left out. However there is no need to truncate, because the computations are readily set up and solved on a spreadsheet. A spreadsheet also enables any sensitivity to assumptions on transition probabilities to be readily examined.

The increments between states need not be constant or symmetrical as in the above examples. States are chosen to suit the particulars of the investment. In drawing transition diagrams, it is important to recognize all possible states and transitions, but having said that, there are no restrictions on the number of states or the number of transitions.

The probability of simultaneous state transitions is assumed small and is excluded. The probability of transitions outside the range of states listed in the transition diagrams is assumed small and is neglected in the above analysis, but could be included. This would only lead to a higher number of states.

In a single paper, it is not possible to demonstrate all possible variations on transition diagrams, but those given are reasonably representative. The transition diagrams represent the investor’s modeling of the investment scenario. The approach is a tool for evaluating an investment’s feasibility.

Transitions between states, which are not adjacent in the above figures, may be possible in practice. For example, in Figure 2, it may be possible to go from a discount rate of 5% to 6% in one transition. For such cases, the extra transitions, with their associated probabilities, can be inserted into the diagrams. Such cases represent no additional formulation time or computations.
State types additional to the three exampled here (discount rate, future cash flow, and investment lifespan) are possible. For example, future cash flows could be broken down into lump sum and series components.

Allowance can be made for plus and minus deviations in states due to different causes, for example changes in future cash flows due to competitors or changes in future cash flows due to consumer behavior could be introduced. This may increase the number of states, and change the connectivity between states.

On an implementation issue of how to estimate transition probabilities, it is suggested that interviews, experience, and knowledge of the industry would be suitable ways to go. These estimates would be based on an understanding of likely interest rate movements in the economy, ability to borrow capital, an understanding of likely investment costs and returns, and the likely investment environment. The estimates will also depend on the time unit or interval chosen, with higher probabilities associated with larger time units.

The adoption of Markov chains above assumes that the transition probabilities are stationary; that is the probability associated with the movement from one state to another is unchangeable. It also assumes that the chain is of first order; that is the state only depends on the immediately previous occupied state. Following Anderson and Goodman (1957), tests of stationarity and order were undertaken by the authors on typical investment data, and for these data at least, stationarity and first order could be shown to be acceptable assumptions. Future research could empirically examine this assumption further.

The formulation given in this paper used Markov chains. That is both the state space and time were discretized. An extension to Markov processes involving continuous time could be done, as could the adoption of semi-Markov processes (where the time between transitions is a random variable, for example Howard, 1964) but it is not believed that the extra computation and complexity is repaid with extra knowledge about an investment.

Generally probabilistic independence is assumed in the above analyses where needed, on the basis that any correlation information is often not available or would be hard to obtain in practice (see for example, Johar et al., 2010).

References


Figures

Figure 1. Example transition diagram.

Figure 2. Fluctuation in interest/discount rate; example.

Figure 3. Fluctuation in discount rate, future cash flow and investment lifespan; example.
Figure 4. Fluctuation in discount rate and future cash flow ($\times 10^3$); example.

Figure 5. Variation in $E[PW]$ ($\times 10^3$) with lifespan.
Figure 6. Part cumulative distribution function for PBP (years), and feasibility plot.

Figure 7. Variation in E[PW] (×10³) with discount rate.
Figure 8. Part cumulative distribution function for IRR, and feasibility plot.
### Table 1. Standard variable housing loan rate changes over a 10-year period

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<th>Month</th>
<th>Rate (%)</th>
<th>Month</th>
<th>Rate (%)</th>
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<td>7.55</td>
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Biographical sketch

DAVID G. CARMICHAEL is Professor of Civil Engineering and former Head of the Department of Engineering Construction and Management at the University of New South Wales. He is a graduate of the Universities of Sydney and Canterbury; a Fellow of the Institution of Engineers, Australia; a Member of the American Society of Civil Engineers; and a former graded arbitrator and mediator. He publishes, teaches, and consults widely in most aspects of project management, construction management, systems engineering, and problem solving. He is known for his leftfield thinking on project and risk management (Project Management Framework, A. A. Balkema, Rotterdam, 2004), and project planning (Project Planning, and Control, Taylor and Francis, London, 2006).